General Relativity

Week 1

General relativity: Theory of gravity

Space & time jointly modeled by M^{3+1}

Equiped with Lorentzian metric g (M, g): Lorentzian manifold

$\rightsquigarrow$ Causal structure

The geometry tells matter how to move, matter dictates how the spacetime is curved

$$Ric - \frac{1}{2} R \cdot g = 8\pi T$$

Aim of this course: Understand this eqn

- Half of the course geometry / half PDEs

Riemannian metric: Modeled on Euclidean inner product space

Lorentzian metric: Lorentzian inner product space

Def: Let V be an $n+1$ dim. vector space

A Lorentzian inner product is a map

$$m : V \times V \longrightarrow \mathbb{R}$$

which is:

1) bilinear
2) symmetric
3) non-degenerate, i.e. if $m(v, w) = 0 \ \forall w \in V \Rightarrow v = 0$
4) Has signature $(1, n)$ (or $- + + + \dots$)

i.e. maximum dimension of subspace $W \subseteq V$ with $m|_W$ positive definite is n.

Example: On $\mathbb{R}^2$: $m_1(v,w) = -v_0 w_0 + v_1 w_2$

$m_2(v,w) = v_0 w_1 + w_1 v_0$

Given a basis $e_0, \dots, e_n$ of V:

The matrix $m_{\alpha\beta} = m(e_\alpha, e_\beta)$ is <u>symmetric</u> and <u>invertible</u> (non-degenerate)

and has 1 negative eigenvalue, n positive

If $X = X^\alpha e_\alpha$, $Y = Y^\beta e_\beta$

$m(X,Y) = m_{\alpha\beta} X^\alpha Y^\beta$ (Einstein summation)

<u>Prop</u>: There exists a basis $(e_0, e_1, \dots, e_n)$ in which

$$m_{00} = -1, \quad m_{ij} = \delta_{ij}, \quad m_{0i} = 0$$

(Convention: Greek letters $\alpha, \beta, \gamma, \dots = 0, 1, \dots, n$
Latin letters $i, j, k, \dots = 1, \dots, n$)

So every (V,m) isometric to $(\mathbb{R}^{n+1}, \eta)$, $\eta = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & \ddots \end{pmatrix}$

~~Def: Let (V,m) be a Lorentzian~~

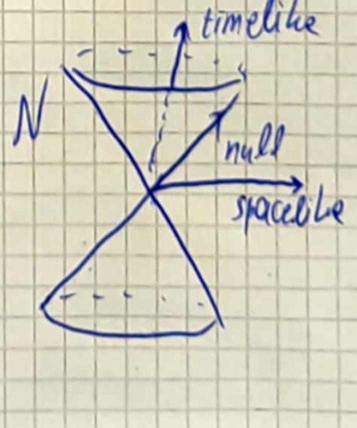
(Inner products of more general signature (p,q): pseudo-Euclidean)

<u>Def</u>: Let (V,m) be a Lorentzian inner product space

A $v \in V/\{0\}$ is

- <u>timelike</u> if $m(v,v) < 0$
- <u>null</u> if $m(v,v) = 0$
- <u>spacelike</u> if $m(v,v) > 0$

$\Big\}$ <u>causal</u>



$v = 0$: spacelike (convention)

Light cone $N = \{v \in V \setminus \{0\} : m(v,v) = 0\}$

Time cone $I = \{v \in V \setminus \{0\} : m(v,v) < 0\}$ ← two components

In the exercises: If v is timelike and $w \perp v \Rightarrow w$ is spacelike

<u>Prop</u>: <u>Gram-Schmidt process</u>: Let v be a timelike vector. We can always find an <u>orthonormal</u> basis $e_0, e_1, \dots, e_n$ (i.e $m(e_\mu, e_\nu) = \eta_{\mu\nu}$) with $e_0 = \lambda \cdot v$, $\lambda > 0$.

<u>Def</u>: A subspace $W \subseteq V$ is called:

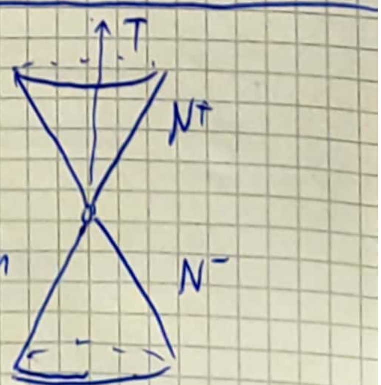
- spacelike if $m|_W$ is positive definite
- null if $m|_W$ is <u>degenerate</u>
- timelike if $m|_W$ is Lorentzian

Most relevant for us: $\dim W = 1$ or $\operatorname{codim} W = 1$.

We will see in the exercises, if $v \in V / \{0\}$:

$$V^\perp = \begin{cases} \text{spacelike}, & \text{if } v \text{ is timelike} \\ \text{null}, & \text{if } v \text{ is null} \quad \text{with } v \in V^\perp \\ \text{timelike}, & \text{if } v \text{ is spacelike} \end{cases}$$

Null cone: N has two components. Pick one, call it <u>future light cone</u> N^+.
Same as picking a timelike direction T: $N^+ = \{v \in N : m(T,v) < 0\}$



Lemma: If v_1, v_2 are timelike vectors in the same time cone, then $m(v_1, v_2) < 0$

Proof: ~~Choose orthonormal basis with e_0~~

Let T be the chosen timelike vector defining the future cone. Choose orthonormal basis $e_0, \dots, e_n$ so that $e_0 = \lambda T$, $\lambda > 0$.

Then v_1, v_2 in the same time cone as T

$$\Rightarrow v_1^0, v_2^0 < 0$$

And v_1 timelike $\Rightarrow (v_1^0)^2 > \sum_{i=1}^{n} (v_1^i)^2$

v_2 " $\Rightarrow (v_2^0)^2 > \sum_{i=1}^{n} (v_2^i)^2$

So: $$v_1^0 v_2^0 \underset{\substack{\uparrow \\ \text{both negative}}}{=} |v_1^0|\,|v_2^0| > \sqrt{\sum_i (v_1^i)^2}\,\sqrt{\sum_i (v_2^i)^2} \underset{\text{Cauchy-Schwarz}}{\geq} \sum_i v_1^i v_2^i$$

$$\Rightarrow \underbrace{-v_1^0 v_2^0 + \sum_{i=1}^{n} v_1^i v_2^i}_{m(v_1, v_2)} < 0$$

∎

Def: $|v| := \sqrt{|m(v,v)|}$

Prop: Let v, w be timelike vectors

1) $|m(v,w)| \geq |v|\,|w|$, equality only if colinear
(reverse Cauchy-Schwartz inequality)

2) If in the same time cone:
$$|v+w| \geq |v| + |w|,$$ equality only if colinear.

3) **Def**: Hyperbolic angle: (if in the same time cone):
$$m(v,w) = -|v|\,|w| \cosh \varphi$$

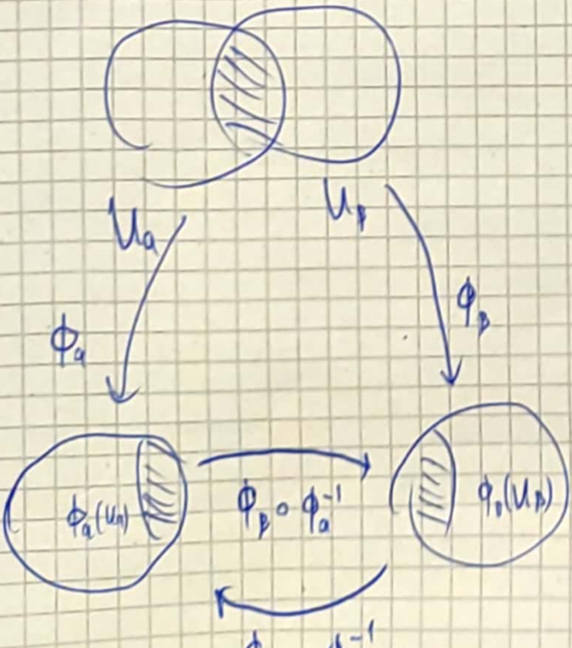
Smooth manifolds

Def: A smooth manifold M of dimension m is a Hausdorff, second countable top. space, together with open sets $\{U_a\}_{a\in A}$ and homeomorphisms $\phi_a : U_a \longrightarrow \phi(U_a) \subseteq \mathbb{R}^m$ s.t.

1) $M = \bigcup_a U_a$

2) If $U_a \cap U_b \neq \emptyset$, then $\phi_a \circ \phi_b^{-1} : \phi_b(U_a \cap U_b) \longrightarrow \phi_a(U_a \cap U_b)$ is a diffeomorphism of domains in $\mathbb{R}^m$

3) (U_a, ϕ_a) Maximal with properties 1 & 2



- (U_a, ϕ_a) : Coordinate chart

  ϕ_a^{-1} : Parametrization

  local coordinates $(x^1, \dots, x^m)$

- A collection $\{(U_a, \phi_a)\}_a$ satisfying 1 & 2: <u>Smooth atlas</u>
- A maximal smooth atlas: <u>Differentiable structure</u>

From the above assumptions: $\exists$ partition of unity

$\{\chi_a\}$
- $\operatorname{supp} \chi_a \subseteq U_a$ for some domain of a coordinate chart
- $\operatorname{supp} \chi_a \cap \operatorname{supp} \chi_b \neq \emptyset$ for finitely many b's
- $\sum_a \chi_a = 1$.

<u>Smooth functions</u>:

<u>Def</u>: Let $F: M \longrightarrow N$ be a function. It is smooth at $p \in M$ (or C^k, etc) if its expression in local coordinates around p and $F(p)$ is smooth (or C^k, etc).

i.e. $\tilde{F} = \psi \circ F \circ \phi^{-1}$ is smooth for any chart (U, ϕ) on M around p and (V, ψ) on N around $F(p)$

<u>Note</u>: Independent of coordinate charts

<u>Tangent space - Cotangent space</u>

Let M be a smooth manifold, $p \in M$.

<u>Def</u>: Tangent vector X_p at p: Linear map $X_p: C^\infty(M) \longrightarrow \mathbb{R}$ such that (Leibniz rule)

$$X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g).$$

T_pM: Set of tangent vectors at p

- Vector space, $\dim T_pM = \dim M$

If $(x^1, \ldots, x^m)$ local coordinates: $\left.\frac{\partial}{\partial x^i}\right|_p$: Coordinate tangent vectors

- Form a basis of T_pM

$\rightarrow$ indices up!

- Any tangent vector: $X = X^i \frac{\partial}{\partial x^i}$

(Smooth) vector field: a choice of a tangent vector X at every $p \in M$ s.t. the components X^i are smooth functions in any local coordinate chart.

Ex: If $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ is a smooth curve:

$\dot{\gamma}(0) \in T_{\gamma(0)}M$, $\quad \dot{\gamma}(0)(f) := \left.\frac{d}{dt}(f \circ \gamma)\right|_{t=0}$

In local coordinates: $\dot{\gamma}^i = \frac{d\gamma^i}{dt}$

Cotangent space : T_p^*M : linear functionals $\omega : T_pM \rightarrow \mathbb{R}$

If $f \in C^\infty(M)$ $\quad df_p(v_p) := \boxed{v_p(f)}$

Basis : $\{dx^i\}_{i=1}^m$

$\uparrow$

Dual basis of $\left\{\frac{\partial}{\partial x^j}\right\}_{j=1}^m$

$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$

$\omega = \omega_i dx^i$

1-Form: ω_i smooth in any local coordinate chart.

Change of coordinate formula:

$$(x^1, \dots, x^n) \longrightarrow (y^1, \dots, y^n)$$

$$\boxed{\begin{aligned} dy^i &= \frac{\partial y^i}{\partial x^j} dx^j \\ \frac{\partial}{\partial y^i} &= \left[\frac{\partial y}{\partial x}^{-1}\right]^j_i \frac{\partial}{\partial x^j} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} \end{aligned}}$$

Lorentzian manifolds:

<u>Def</u>: Let M be a smooth manifold. A <u>Lorentzian metric</u> on M is an assignment g_p of a Lorentzian inner product on T_pM $\forall p \in M$, s.t $g_{\alpha\beta} = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$ is a <u>smooth</u> function in any local coordinate chart.

↑ Symmetric $(0,2)$ tensor field.

$$g(X, Y) = g_{\alpha\beta} X^\alpha Y^\beta$$

We will write $g = g_{\alpha\beta}\, dx^\alpha dx^\beta$

- A tangent vector $X_p \in T_pM$ is timelike / null / spacelike according to our previous definition
- A vector field $X \in \Gamma(M)$ is timelike / null / spacelike if it is " / " / " at every point
- A C^1 curve $\gamma : (a,b) \longrightarrow M$ is timelike / null / spacelike if $\dot{\gamma}$ is
- A submanifold $S \subseteq M$ is similarly ... if $\forall p \in S$, $T_pS \subseteq T_pM$ ...